A Dolbeault-type Double Complex on Quaternionic Manifolds

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Abstract

It has long been known that differential forms on a complex manifold M^{2n} can be decomposed under the action of the complex structure to give the Dolbeault complex. This paper presents an analogous double complex for a quaternionic manifold M^{4n} using the fact that its cotangent space T_m^*M is isomorphic to the quaternionic vector space $\mathbb{H}^n$. This defines an action of the group $\mathrm{Sp}(1)$ of unit quaternions on T^*M, which induces an action of $\mathrm{Sp}(1)$ on the space of k-forms $\Lambda^k T^*M$. A double complex is obtained by decomposing $\Lambda^k T^*M$ into irreducible representations of $\mathrm{Sp}(1)$, resulting in new 'quaternionic Dolbeault' operators and cohomology groups.

Links with previous work in quaternionic geometry, particularly the differential complex of Salamon and the q-holomorphic functions of Joyce, are demonstrated.

Introduction

This paper describes a new double complex of differential forms on hypercomplex or quaternionic manifolds. This is to date the clearest quaternionic version of the more familiar Dolbeault complex, used throughout complex geometry. It is hoped that readers familiar with complex geometry will find the new ideas both natural and familiar. An important thread will be to examine the role played by the group $\mathrm{U}(1)$ of unit complex numbers in complex geometry: complex geometry is then translated into quaternionic geometry by replacing $\mathrm{U}(1)$ with the group $\mathrm{Sp}(1)$ of unit quaternions. This approach is rewarding because of the simplicity of the representations of the groups $\mathrm{U}(1)$ and $\mathrm{Sp}(1)$, which makes their action much easier to understand than those of the groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$.

The paper is arranged as follows. Section 1 contains background material from complex and quaternionic geometry, reviewing the definition of a complex manifold and considering quaternionic analogues. At least two definitions are currently recognised. Following the work of Salamon [S1], the term 'quaternionic manifold' is used for a manifold possessing a torsion-free $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$-structure. The more restricted class of 'hypercomplex manifolds' refers to those with a torsion-free $\mathrm{GL}(n,\mathbb{H})$-structure. Hypercomplex manifolds possess global anticommuting complex structures (since $\mathrm{GL}(n,\mathbb{H}) \subset \mathrm{GL}(2n,\mathbb{C})$). Quaternionic manifolds possess a 2-sphere bundle of local anti-commuting almost-complex structures, but whilst this bundle is invariant it may have no

global sections, so there may be no global complex structures. The Riemannian versions of these structures (Kähler, hyperkähler and quaternionic Kähler manifolds) are briefly described. This section also recalls the most basic facts about Sp(1)-representations, their weights and tensor products, which we later use to decompose forms on quaternionic manifolds. Associated bundles are described.

Section 2 reviews the decomposition of differential forms in complex geometry, and the resulting Dolbeault complex. (This material will be familiar to most readers.) The space $\Lambda^{p,q}M$ of (p,q)-forms on a complex manifold (M,I) is a representation of the Lie algebra $\mathfrak{u}(1)$ via the induced action of the complex structure I on $\Lambda^k T^*M$, a point of view which adapts well to quaternionic geometry. As well as the standard decomposition of complex-valued forms, we also describe the less familiar decomposition of real-valued forms. The resulting 'real Dolbeault complex' is even more closely akin to the new quaternionic complex because they both form isosceles triangles (as opposed to the diamond configuration of the standard Dolbeault complex).

Section 3 surveys previous work on the decomposition of differential forms in quaternionic geometry. Kraines [K] and Bonan [B] obtained a decomposition of forms on quaternionic Kähler manifolds by taking successive exterior products with the fundamental 4-form. Swann [Sw] considered decomposition of these forms as Sp(1)Sp(n)-representations. In the non-Riemannian setting, Salamon [S1] used the coarser decomposition of forms on quaternionic manifolds into Sp(1)GL($n,\mathbb{H}$)-representations, resulting in a differential complex which forms the top row of the new double complex. Much of this algebra and geometry can be inferred from Fujiki's comprehensive article [F], which describes much of the theory underlying this whole area of research.

The heart of this paper is Section 4 which constructs the analogue of the Dolbeault complex in quaternionic geometry. The complex structure I is replaced by the (possibly local) almost complex structures I, J and K, the group U(1) of unit complex numbers is replaced by the group Sp(1) of unit quaternions, and the Lie algebra $\mathfrak{u}(1) = \langle I \rangle$ is replaced by $\mathfrak{sp}(1) = \langle I, J, K \rangle$. Despite the possible lack of global complex structures on quaternionic manifolds, the Casimir operator $I^2 + J^2 + K^2$ still makes invariant sense. It follows that the decomposition of $\Lambda^k T^*M$ into irreducible Sp(1)-representations is also invariant. A straightforward calculation using weights leads to the following result (Proposition 4.1):

$$\Lambda^k T^*M \cong \bigoplus_{r=0}^{k} \left[\binom{2n}{\frac{k+r}{2}} \binom{2n}{\frac{k-r}{2}} - \binom{2n}{\frac{k+r+2}{2}} \binom{2n}{\frac{k-r-2}{2}} \right] V_r,$$

where M^{4n} is a quaternionic manifold, V_r is the Sp(1)-representation with highest weight r, and $r \equiv k \bmod 2$. It follows from the Clebsch-Gordon formula $V_r \otimes V_1 \cong V_{r+1} \oplus V_{r-1}$ that the exterior derivative of a form in the V_r component of $\Lambda^k T^*M$ has components only in the V_{r+1} and V_{r-1} components of $\Lambda^{k+1}T^*M$. This demonstrates with reassuring simplicity that this decomposition naturally leads to a new double complex. As with the Dolbeault complex, the exterior differential $d\omega$ of a k-form ω can be split up into the components $\mathcal{D}\omega$ and $\overline{\mathcal{D}}\omega$, which are the components of $d\omega$ lying in Sp(1)-representations of higher and lower weight respectively. We describe the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ and define new cohomology groups.

In Section 5 we determine where the (upward) complex is elliptic. This proves to be a tricky problem, requiring further decomposition and careful analysis. Fortunately, this

hard work proves to be well worthwhile, as the double complex is shown to be elliptic in most places. Like the real Dolbeault complex, ellipticity only fails at the bottom of the isosceles triangle of spaces where $\overline{\mathcal{D}} = 0$ and $\mathcal{D} = d$, breaking the usual pattern which relies on studying non-trivial projection maps.

Finally, in Section 6 we consider some extra opportunities for decomposition of quaternion-valued forms in the more restricted class of hypercomplex manifolds. Here the global complex structures I, J and K can be identified with the action of the imaginary quaternions i, j and k on $\mathbb{H}^n$. This allows a further decomposition of quaternion-valued forms, which can be used to develop the theory of quaternionic analysis on hypercomplex manifolds. For example, Joyce's algebraic theory of quaternion holomorphic functions [J] can be described using this type of decomposition.

1 Background Material

This section contains background information in complex and quaternionic geometry, reviewing the definition of a complex manifold and the ways in which this is adapted to quaternions. We also recall some basic facts concerning the algebra of Sp(1)-representations.

1.1 Complex, Hypercomplex and Quaternionic Manifolds

An *almost complex structure* on a $2n$-dimensional real manifold M is a smooth tensor $I \in C^\infty(\mathrm{End}(TM))$ such that $I^2 = -\operatorname{id}_{TM}$. An almost complex structure I is said to be integrable if and only if the Nijenhuis tensor of I

$$N_I(X,Y) = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY]$$

vanishes for all $X, Y \in C^\infty(TM)$, for all $x \in M$. In this case it can be proved that the almost complex structure I arises from a suitable set of holomorphic coordinates on M, and the pair (M, I) is said to be a complex manifold. This way of defining a complex manifold adapts itself well to the quaternions.

Definition 1.1 An *almost hypercomplex structure* on a $4n$-dimensional manifold M is a triple (I, J, K) of almost complex structures on M which satisfy the relation $IJ = K$. An almost hypercomplex structure on M defines an isomorphism $T_xM \cong \mathbb{H}^n$ at each point $x \in M$.

If all of the complex structures are integrable then (I, J, K) is called a *hypercomplex structure* on M, and M is a *hypercomplex manifold*.

Not all of the manifolds which we wish to describe as 'quaternionic' admit hypercomplex structures. For example, the quaternionic projective line $\mathbb{H}P^1$ is diffeomorphic to the 4-sphere S^4. It is well-known that S^4 does not even admit a global almost complex structure; so $\mathbb{H}P^1$ can certainly not be hypercomplex, despite behaving extremely like the quaternions locally.

The reason (and the solution) for this difficulty can be described succinctly in terms of G-structures on manifolds. Let P be the principal frame bundle of M, *i.e.* the $\mathrm{GL}(n, \mathbb{R})$-bundle whose fibre over $x \in M$ is the group of isomorphisms $T_xM \cong \mathbb{R}^{4n}$. Let G be a

Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. A *$G$-structure* Q on M is a principal subbundle of P with structure group G.

Suppose M^{2n} has an almost complex structure. The group of automorphisms of $T_x M$ preserving such a structure is isomorphic to $\mathrm{GL}(n, \mathbb{C})$. Thus an almost complex structure I and a $\mathrm{GL}(n, \mathbb{C})$-structure Q on M contain the same information. The bundle Q admits a torsion-free connection if and only if there is a torsion-free linear connection ∇ on M with $\nabla I = 0$, in which case it is easy to show that I is integrable. Thus a complex manifold is precisely a real manifold M^{2n} with a $\mathrm{GL}(n, \mathbb{C})$-structure Q admitting a torsion-free connection (in which case Q itself is said to be 'integrable').

A hypercomplex manifold is a real manifold M with an integrable $\mathrm{GL}(n, \mathbb{H})$-structure Q. However, the group $\mathrm{GL}(n, \mathbb{H})$ is not the largest subgroup of $\mathrm{GL}(4n, \mathbb{R})$ preserving the quaternionic structure of $\mathbb{H}^n$. If $\mathrm{GL}(n, \mathbb{H})$ acts on $\mathbb{H}^n$ by right-multiplication by $n \times n$ quaternionic matrices, then the action of $\mathrm{GL}(n, \mathbb{H})$ commutes with that of the left $\mathbb{H}$-action of the group $\mathrm{GL}(1, \mathbb{H}) \cong \mathbb{H}^*$. Thus the group of symmetries of $\mathbb{H}^n$ is the product $\mathrm{GL}(1, \mathbb{H}) \times_{\mathbb{R}^*} \mathrm{GL}(n, \mathbb{H})$. Scaling the first factor by a real multiple of the identity reduces the first factor to $\mathrm{Sp}(1)$, and $\mathrm{GL}(1, \mathbb{H}) \times_{\mathbb{R}^*} \mathrm{GL}(n, \mathbb{H})$ is the same as $\mathrm{Sp}(1) \times_{\mathbb{Z}_2} \mathrm{GL}(n, \mathbb{H})$ which is normally abbreviated to $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$.

Definition 1.2 [S1, 1.1] A *quaternionic manifold* is a $4n$-dimensional real manifold M ($n \geq 2$) with an $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure Q admitting a torsion-free connection.

When $n = 1$ the situation is different, since $\mathrm{Sp}(1)\mathrm{Sp}(1) \cong \mathrm{SO}(4)$. In four dimensions we make the special definition that a quaternionic manifold is a self-dual conformal manifold.

In terms of tensors, quaternionic manifolds are a generalisation of hypercomplex manifolds in the following way. Each tangent space $T_x M$ still admits a hypercomplex structure giving an isomorphism $T_x M \cong \mathbb{H}^n$, but this isomorphism does not necessarily arise from globally defined complex structures on M. There is still an invariant S^2-bundle of local almost-complex structures satisfying the equation $IJ = K$, but it is free to 'rotate'.

The manifolds defined above all have Riemannian counterparts defined by a reduction of the structure group G to a compact subgroup. A complex manifold M whose $\mathrm{GL}(n, \mathbb{C})$-structure reduces to an integrable $\mathrm{U}(n)$-structure admits a Riemannian metric g with $g(IX, IY) = g(X, Y)$ for all $X, Y \in T_x M$ for all $x \in M$. In this case M is called a Kähler manifold and the metric g is called a Kähler metric. The differentiable 2-form $\omega(X, Y) = g(IX, Y)$, called the Kähler form, is closed and M is a symplectic manifold — so M has compatible complex and symplectic structures.

The quaternionic analogue of the unitary group $\mathrm{U}(n)$ is the compact group $\mathrm{Sp}(n)$. A hypercomplex manifold whose $\mathrm{GL}(n, \mathbb{H})$-structure Q reduces to an integrable $\mathrm{Sp}(n)$-structure Q' admits a metric g which is simultaneously Kähler for each of the complex structures I, J and K. Such manifolds are called *hyperkähler* and have been extensively studied, Fujiki's account [F] being perhaps the most relevant in this context. Hyperkähler manifolds have three independent symplectic forms ω_I, ω_J and ω_K. The complex 2-form $\omega_J + i\omega_K$ is holomorphic with respect to the complex structure I, and a hyperkähler manifold has compatible hypercomplex and complex-symplectic structures.

Similarly, if a quaternionic manifold has a metric compatible with the torsion-free $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure, then the $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure Q reduces to an $\mathrm{Sp}(1)\mathrm{Sp}(n)$-

structure Q' and M is said to be *quaternionic Kähler*. The group $\mathrm{Sp}(1)\mathrm{Sp}(n)$ is a maximal proper subgroup of $\mathrm{SO}(4n)$ except when $n = 1$, where $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$. In four dimensions a manifold is said to be quaternionic Kähler if and only if it is self-dual and Einstein.

1.2 Sp(1)-Representations

Let $\mathrm{Sp}(1)$ be the multiplicative group of unit quaternions. Its Lie algebra $\mathfrak{sp}(1)$ is generated by I, J and K and the bracket relations $[I, J] = 2K$, $[J, K] = 2I$ and $[K, I] = 2J$. The representations of $\mathrm{Sp}(1)$ are ubiquitous in modern mathematics, often in the guise of representations of the isomorphic group $\mathrm{SU}(2)$ or the complexification $\mathrm{SL}(2, \mathbb{C})$. Standard texts include [BD, §2.5] and [FH, Lecture 11]. The second of these is particularly useful for describing $\mathrm{Sp}(1)$-representations and their tensor products using weights of the action of $\mathfrak{sl}(2, \mathbb{C})$. This technique is instrumental in decomposing exterior forms on quaternionic manifolds. We recall the most salient points.

Following standard practice we work primarily with representations on complex vector spaces, real (and quaternionic) representations being constructed in the presence of suitable structure maps. Every representation of $\mathrm{Sp}(1)$ on a complex vector space can be written as a direct sum of irreducible representations and the multiplicity of each irreducible in such a decomposition is uniquely determined.

Let V_1 be the basic representation of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ on $\mathbb{C}^2$ given by left-action of matrices upon column vectors. The n^{th} symmetric power of V_1 is a representation on $\mathbb{C}^{n+1}$ which is written

$$V_n = S^n(V_1).$$

The representation V_n is irreducible and every irreducible representation of $\mathrm{Sp}(1)$ is of the form V_n for some nonnegative $n \in \mathbb{Z}$. Each irreducible representation V_n is an eigenspace of the Casimir operator $I^2 + J^2 + K^2$ with eigenvalue $-n(n + 2)$.

The irreducible representation V_n can be decomposed into weight spaces under the action of a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. Each weight space is one-dimensional and the weights are the integers

$$\{n, n - 2, \ldots, n - 2k, \ldots, 2 - n, -n\}.$$

Thus V_n is also characterised by being the unique irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight n.

It follows from the Leibniz rule $I(a \otimes b) = I(a) \otimes b + a \otimes I(b)$ for Lie algebra representations that if $a \in V_m$ and $b \in V_n$ are weight vectors of I with weights λ and μ then $a \otimes b$ is a weight vector of $V_m \otimes V_n$ with weight $\lambda + \mu$. Weight space decompositions can thus be used to determine tensor, symmetric and exterior products of $\mathrm{Sp}(1)$-representations. Amongst other things, this enables us to calculate the irreducible decomposition of the (diagonal) action of $\mathrm{Sp}(1)$ on the tensor product $V_m \otimes V_n$. This is given by the famous *Clebsch-Gordon formula*,

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n+2} \oplus V_{m-n} \qquad \text{for } m \geq n. \quad (1)$$

1.3 Associated Bundles

It is usual in differential geometry to talk about properties of a vector bundle using the properties of its fibres. A good example of this is when talking about a group acting on a vector bundle: or more precisely, a principal bundle whose fibres act upon the fibres of an associated vector bundle.

Let P be a principal G-bundle over the differentiable manifold M and let V be a representation of the group G. We define the *associated bundle*

$$\mathbf{V} = P \times_G V = \frac{P \times V}{G},$$

where $g \in G$ acts on $(p, v) \in P \times V$ by $(p, v) \cdot g = (f \cdot g, g^{-1} \cdot v)$. Then $\mathbf{V}$ is a vector bundle over M with fibre V. At every point $m \in M$ the fibre $P_m \cong G$ acts on the fibre $\mathbf{V}_m$, a notion that is commonly abused slightly by saying that G acts on $\mathbf{V}$. A decomposition of V into subrepresentations then gives rise to a decomposition of $\mathbf{V}$ into associated subbundles. We will usually just write V for $\mathbf{V}$, relying on context to distinguish between the bundle and the representation.

In this way the cotangent spaces of complex and quaternionic manifolds are described using representations of the groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$ respectively: or indeed using the much simpler representations of the groups $\mathrm{U}(1)$ and $\mathrm{Sp}(1)$.

2 Differential Forms on Complex Manifolds

Let (M, I) be a complex manifold. The complexified cotangent space splits into eigenspaces of I with eigenvalues $\pm i$, $T^*M^{\mathbb{C}} = T^*_{1,0}M \oplus T^*_{0,1}M$, which are called the holomorphic and antiholomorphic cotangent spaces respectively. This induces the familiar decomposition into types of exterior k-forms

$$\Lambda^k T^*M^{\mathbb{C}} = \bigoplus_{p+q=k} \Lambda^p(T^*_{1,0}M) \otimes \Lambda^q(T^*_{0,1}M),$$

where the bundle $\Lambda^{p,q}M = \Lambda^p T^*_{1,0}M \otimes \Lambda^q T^*_{0,1}M$ is called the bundle of (p, q)-forms on M. A smooth section of the bundle $\Lambda^{p,q}M$ is called a differential form of type (p, q) or just a (p, q)-form. We write $\Omega^{p,q}(M)$ for the set of (p, q)-forms on M, so

$$\Omega^{p,q}(M) = C^\infty(M, \Lambda^{p,q}M) \qquad \text{and} \qquad \Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Define two first-order differential operators,

$$\begin{array}{ll} \partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \\ \partial = \pi^{p+1,q} \circ d \end{array} \qquad \text{and} \qquad \begin{array}{ll} \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M) \\ \overline{\partial} = \pi^{p,q+1} \circ d, \end{array} \tag{2}$$

where $\pi^{p,q}$ denotes the natural projection from $\Lambda^k T^*M^{\mathbb{C}}$ onto $\Lambda^{p,q}M$. The operators ∂ and $\overline{\partial}$ are called the Dolbeault operators.

These definitions rely only on the fact that I is an almost complex structure. If in addition I is integrable then these are the only two components of the exterior differential

Figure 1: The Dolbeault Complex

$$
\begin{tikzcd}
 & & \Omega^{2,0} \arrow[dr, "\overline{\partial}"] & \ldots\ldots\ \textit{etc.} \\
 & \Omega^{1,0} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"] & & {} \\
\Omega^{0,0} = C^\infty(M, \mathbb{C}) \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"] & & \Omega^{1,1} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"] & \\
 & \Omega^{0,1} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"] & & {} \\
 & & \Omega^{0,2} \arrow[ur, "\partial"] & \ldots\ldots\ \textit{etc.}
\end{tikzcd}
$$

d, so that $d = \partial + \overline{\partial}$ [W, p.34]. An immediate consequence of this is that on a complex manifold M, $\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0$. This gives rise to the *Dolbeault complex*.

The purpose of this paper is precisely to present the quaternionic analogue of this double complex. To do this, note that the bundle $\Lambda^{p,q}M$ is an eigenspace of the induced action of I on $\Lambda^k T^*M^{\mathbb{C}}$, since for $\omega \in \Lambda^{p,q}M$, $I(\omega) = i(p-q)\omega$. The decomposition into types can therefore be thought of as a decomposition of $\Lambda^k T^*M^{\mathbb{C}}$ into $\mathfrak{u}(1)$-representations, where the complex structure I generates a copy of the Lie algebra $\mathfrak{u}(1)$. [1] We will presently do this in quaternionic geometry by replacing the Lie algebra $\mathfrak{u}(1) = \langle I \rangle$ with $\mathfrak{sp}(1) = \langle I, J, K \rangle$ and decompose $\Lambda^k T^*M$ into irreducible $\mathfrak{sp}(1)$-representations.

2.1 Real forms on Complex Manifolds

It is less well-known that a similar splitting occurs for real-valued exterior forms. This is an instructive case, because the resulting double complex is even more closely akin to the new quaternionic double complex.

Let M be a complex manifold and let $\omega \in \Lambda^{p,q} = \Lambda^{p,q}M$. For simplicity's sake assume that $p > q$ throughout. Then $\overline{\omega} \in \Lambda^{q,p}$, and $\omega + \overline{\omega}$ is a real-valued exterior form. Define the space of such forms,

$$(\Lambda^{p,q} \oplus \Lambda^{q,p})_{\mathbb{R}} = [\Lambda^{p,q} \oplus \Lambda^{q,p}] \equiv [[\Lambda^{p,q}]].$$

The space $[[\Lambda^{p,q}]]$ is a real vector bundle associated to the principal $\mathrm{GL}(n, \mathbb{C})$-bundle defined by the complex structure. The first square bracket indicates real forms and the second the direct sum, the notation following that of Reyes-Carrión [R, §3.1], who uses the ensuing decomposition on Kähler manifolds.

[1] The irreducible representations of $\mathrm{U}(1)$ are all one-dimensional. They are parametrised by the integers, taking the form $\varrho_n : \mathrm{U}(1) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\varrho_n : e^{i\theta} \to e^{ni\theta}$ for some $n \in \mathbb{Z}$. The corresponding representations of the Lie algebra $\mathfrak{u}(1)$ are then of the form $d\varrho_n : z \to inz$.

This gives a decomposition of real-valued exterior forms,

$$\Lambda^k_{\mathbb{R}} T^*M = \bigoplus_{\substack{p+q=k \\ p>q}} [[\Lambda^{p,q}]] \oplus [\Lambda^{\frac{k}{2},\frac{k}{2}}]. \tag{3}$$

The condition $p > q$ ensures that we have no repetition. The bundle $[\Lambda^{\frac{k}{2},\frac{k}{2}}]$ only appears when k is even. It is its own conjugate and so naturally a real vector bundle associated to the trivial representation (*i.e.* the zero weight space) of the Lie algebra $\mathfrak{u}(1) = \langle I \rangle$.

Let $[[\Omega^{p,q}]] = C^\infty([[\Lambda^{p,q}]])$, so that for $\omega \in \Omega^{p,q}(M)$, $\omega + \overline{\omega} \in [[\Omega^{p,q}]]$. Then

$$d(\omega + \overline{\omega}) = (\partial\omega + \overline{\partial}\overline{\omega}) + (\overline{\partial}\omega + \partial\overline{\omega}) \in [[\Omega^{p+1,q}]] \oplus [[\Omega^{p,q+1}]].$$

Call the first of these components $[\partial]\omega$ and the second $[\overline{\partial}]\omega$. This defines real analogues of the Dolbeault operators. The 'double complex' equations $[\partial]^2 = [\partial][\overline{\partial}] + [\overline{\partial}][\partial] = [\overline{\partial}]^2 = 0$ follow directly from decomposing the equation $d^2 = 0$. For the space $[\Lambda^{\frac{k}{2},\frac{k}{2}}]$ there is no space $[[\Lambda^{\frac{k}{2},\frac{k}{2}+1}]]$ because $p > q$, so $[\overline{\partial}] = 0$ and there is just one operator $[\partial] = d$.

Figure 2: The Real Dolbeault Complex

$$\begin{tikzcd}
 & & & {[[\Omega^{3,0}]]} \arrow[dr, "{[\overline{\partial}]}"] & & \ldots\ldots\, etc. \\
 & & {[[\Omega^{2,0}]]} \arrow[ur, "{[\partial]}"] \arrow[dr, "{[\overline{\partial}]}"] & & {} & \\
 & {[[\Omega^{1,0}]] = \Omega^1(M)} \arrow[ur, "{[\partial]}"] \arrow[dr, "{[\overline{\partial}]}"] & & {[[\Omega^{2,1}]]} \arrow[ur, "{[\partial]}"] \arrow[dr, "{[\overline{\partial}]}"] & & \ldots\ldots\, etc. \\
{[\Omega^{0,0}] = C^\infty(M)} \arrow[ur, "d"] & & {[\Omega^{1,1}]} \arrow[ur, "d"] & & {} &
\end{tikzcd}$$

Thus there is a double complex of real forms on a complex manifold, obtained by decomposing $\Lambda^k_x T^*M$ into subrepresentations of the action of $\mathfrak{u}(1) = \langle I \rangle$, induced from the action on $T^*_x M$. The main difference is that this real-valued complex gives an isosceles triangle of spaces, whereas the standard Dolbeault complex gives a full diamond. Amalgamating the spaces $\Lambda^{p,q}$ and $\Lambda^{q,p}$ into the single real space $[[\Lambda^{p,q}]]$ has effectively folded this diamond in half. This structure is very similar to that of the new quaternionic double complex which is the main subject of this paper.

Ellipticity

A complex $0 \xrightarrow{\Phi_0} C^\infty(E_0) \xrightarrow{\Phi_1} C^\infty(E_1) \xrightarrow{\Phi_2} C^\infty(E_2) \xrightarrow{\Phi_3} \ldots \xrightarrow{\Phi_n} C^\infty(E_n) \xrightarrow{\Phi_{n+1}} 0$ is said to be elliptic at E_i if the *principal symbol sequence* $E_{i-1} \xrightarrow{\sigma_{\Phi_i}} E_i \xrightarrow{\sigma_{\Phi_{i+1}}} E_{i+1}$ is exact for all

$\xi \in T_x^*M$ and for all $x \in M$. [2] A thorough description of this topic can be found in [W, Chapter 5]. For our operators, it suffices to note that principal symbol of the exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is $\sigma_d(x,\xi)\omega = \omega \wedge \xi$. Let $\pi : \Lambda^k T^*M \to E$ be the projection from $\Lambda^k T^*M$ onto some subspace E. It follows that the principal symbol of $\pi \circ d$ on $\Lambda^{k-1}T^*M$ is just $\sigma_{\pi\circ d}(x,\xi)\omega = \pi(\omega \wedge \xi)$.

It is important to establish where a differential complex is elliptic for various reasons: for example, an elliptic complex on a compact manifold always has finite-dimensional cohomology groups [W, Theorem 5.2, p. 147]. The de Rham and Dolbeault complexes are elliptic everywhere. The real Dolbeault complex of Figure 2 is elliptic in most places, but not everywhere. Interestingly, it fails to be elliptic in almost exactly the same places as the new quaternionic double complex, and for the same reasons.

Proposition 2.1 *For $p > 0$, the upward complex*

$$0 \longrightarrow [\Omega^{p,p}] \xrightarrow{d} [[\Omega^{p+1,p}]] \xrightarrow{[\partial]} [[\Omega^{p+2,p}]] \xrightarrow{[\partial]} \ldots$$

is elliptic everywhere except at the first two spaces $[\Omega^{p,p}]$ and $[[\Omega^{p+1,p}]]$.

For $p = 0$, the 'leading edge' complex

$$0 \longrightarrow [\Omega^{0,0}] \xrightarrow{d} [[\Omega^{1,0}]] \xrightarrow{[\partial]} [[\Omega^{2,0}]] \xrightarrow{[\partial]} \ldots$$

is elliptic everywhere except at $[[\Omega^{1,0}]] = \Omega^1(M)$.

Proof. When $p > q+1$, the short sequence $[[\Omega^{p-1,q}]] \xrightarrow{[\partial]} [[\Omega^{p,q}]] \xrightarrow{[\partial]} [[\Omega^{p+1,q}]]$ is a real form of the sequence

$$\begin{array}{ccccc} \Omega^{p-1,q} & \xrightarrow{\partial} & \Omega^{p,q} & \xrightarrow{\partial} & \Omega^{p+1,q} \\ \bigoplus & & \bigoplus & & \bigoplus \\ \Omega^{q,p-1} & \xrightarrow{\bar\partial} & \Omega^{q,p} & \xrightarrow{\bar\partial} & \Omega^{q,p+1}. \end{array}$$

This is (a real subspace of) the direct sum of two elliptic sequences, and so is elliptic. Thus we have ellipticity at $[[\Omega^{p,q}]]$ whenever $p \geq q+2$.

This leaves us to consider the case when $p = q$, giving (a real subspace of) the sequence

$$\begin{array}{cccccccc} & & & \overset{\partial}{\nearrow} & \Omega^{p+1,p} & \xrightarrow{\partial} & \Omega^{p+2,2} & \longrightarrow \ldots etc. \\ 0 & \longrightarrow & \Omega^{p,p} & & \bigoplus & & \bigoplus & \\ & & & \underset{\bar\partial}{\searrow} & \Omega^{p,p+1} & \xrightarrow{\bar\partial} & \Omega^{2,p+2} & \longrightarrow \ldots etc. \end{array} \tag{4}$$

This fails to be elliptic. An easy and instructive way to see this is to consider the simplest 4-dimensional example $M = \mathbb{C}^2$.

Let e^0, $e^1 = I(e^0)$, e^2 and $e^3 = I(e^2)$ form a basis for $T_x^*\mathbb{C}^2 \cong \mathbb{C}^2$, and let $e^{ab\ldots}$ denote $e^a \wedge e^b \wedge \ldots etc.$ Then $I(e^{01}) = e^{00} - e^{11} = 0$, so $e^{01} \in [\Lambda^{1,1}]$. The map from $[\Lambda^{1,1}]$ to $[[\Lambda^{2,1}]]$ is just the exterior differential d. Since $\sigma_d(x,e^0)(e^{01}) = e^{01} \wedge e^0 = 0$ the symbol map $\sigma_d : [\Lambda^{1,1}] \to [[\Lambda^{2,1}]]$ is not injective, so the symbol sequence is not exact at $[\Lambda^{1,1}]$.

[2] The link between elliptic complexes and elliptic operators (those whose principal symbol is an isomorphism) is as follows. Given any metric on each E_i, define a formal adjoint $\Phi_i^* : E_i \to E_{i-1}$. The complex is elliptic at E_i if and only if the Laplacian $\Phi_i^*\Phi_i + \Phi_{i-1}\Phi_{i-1}^*$ is an elliptic operator.

Consider also $e^{123} \in [[\Lambda^{2,1}]]$. Then $\sigma_{[\partial]}(x, e^0)(e^{123}) = 0$, since there is no bundle $[[\Lambda^{3,1}]]$. But e^{123} has no e^0-factor, so is not the image under $\sigma_d(x, e^0)$ of any form $\alpha \in [\Lambda^{1,1}]$. Thus the symbol sequence fails to be exact at $[[\Lambda^{2,1}]]$.

It is a simple matter to extend these counterexamples to higher dimensions and higher exterior powers. For $k = 0$, the situation is different. It is easy to show that the complex

$$0 \longrightarrow C^\infty(M) \xrightarrow{d} [[\Omega^{1,0}]] \xrightarrow{[\partial]} [[\Omega^{2,0}]] \longrightarrow \ldots \textit{etc.}$$

is elliptic everywhere except at $[[\Omega^{1,0}]]$. ∎

This last sequence is given particular attention by Reyes-Carrión [R, Lemma 2]. He shows that, when M is Kähler, ellipticity can be regained by adding the space $\langle \omega \rangle$ to the bundle $[[\Lambda^{2,0}]]$, where ω is the real Kähler $(1,1)$-form.

The real Dolbeault complex is thus elliptic except at the bottom of the isosceles triangle of spaces. Here the projection from $d([\Omega^{p,p}])$ to $[[\Omega^{p+1,p}]]$ is the identity, and arguments based upon non-trivial projection maps no longer apply. We shall see that this situation is closely akin to that of differential forms on quaternionic manifolds, and that techniques motivated by this example yield similar results.

3 Differential Forms on Quaternionic Manifolds

This section describes previous results in the decomposition of exterior forms in quaternionic geometry. These fall into two categories: those arising from taking repeated products with the fundamental 4-form in quaternionic Kähler geometry and those arising from considering the representations of $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$ on $\Lambda^k T^*M$. We are primarily concerned with the second approach.

The decomposition of differential forms on quaternionic Kähler manifolds began by considering the *fundamental* 4-*form*

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K,$$

where ω_I, ω_J and ω_K are the local Kähler forms associated to local almost complex structures I, J and K with $IJ = K$. The fundamental 4-form is globally defined and invariant under the induced action of Sp(1)Sp(n) on $\Lambda^4 T^*M$. Kraines [K] and Bonan [B] used the fundamental 4-form to decompose the space $\Lambda^k T^*M$ in a similar way to the Lefschetz decomposition of differential forms on a Kähler manifold [GH, p. 122]. A differential k-form μ is said to be *effective* if $\Omega \wedge *\mu = 0$, where $* : \Lambda^k T^*M \to \Lambda^{4n-k} T^*M$ is the Hodge star. This leads to the following theorem:

Theorem 3.1 *[K, Theorem 3.5][B, Theorem 2] Let M^{4n} be a quaternionic Kähler manifold. For $k \leq 2n+2$, every every k-form ϕ admits a unique decomposition*

$$\phi = \sum_{0 \leq j \leq k/4} \Omega^j \wedge \mu_{k-4j},$$

where the μ_{k-4j} are effective $(k-4j)$-forms.

Bonan further refines this decomposition for quaternion-valued forms, using exterior multiplication by the globally defined quaternionic 2-form $\Psi = i_1\omega_I + i_2\omega_J + i_3\omega_K$. Note that $\Psi \wedge \Psi = -2\Omega$.

Another way to consider the decomposition of forms on a quaternionic manifold is as subbundles of $\Lambda^k T^*M$ associated with different representations of the group $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$. The representation of $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$ on $\mathbb{H}^n$ is given by the equation

$$\mathbb{H}^n \otimes_{\mathbb{R}} \mathbb{C} \cong V_1 \otimes E, \tag{5}$$

where V_1 is the basic representation of $\mathrm{Sp}(1)$ on $\mathbb{C}^2$ and E is the basic representation of $\mathrm{GL}(n,\mathbb{H})$ on $\mathbb{C}^{2n}$. (This uses the standard convention of working with complex representations, which in the presence of suitable structure maps can be thought of as complexified real representations. In this case, the structure map is the tensor product of the quaternionic structures on V_1 and E.)

This representation also describes the (co)tangent bundle of a quaternionic manifold in the following way. Following Salamon [S1, §1], if M^{4n} is a quaternionic manifold with $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$-structure Q, then the cotangent bundle is a vector bundle associated with the principal bundle Q and the representation $V_1 \otimes E$, so that

$$(T^*M)^{\mathbb{C}} \cong V_1 \otimes E \tag{6}$$

(though we will usually omit the complexification sign). This induces an $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$-action on the bundle of exterior k-forms $\Lambda^k T^*M$,

$$\Lambda^k T^*M \cong \Lambda^k(V_1 \otimes E) \cong \bigoplus_{j=0}^{[k/2]} S^{k-2j}(V_1) \otimes L_j^k \cong \bigoplus_{j=0}^{[k/2]} V_{k-2j} \otimes L_j^k, \tag{7}$$

where L_j^k is an irreducible representation of $\mathrm{GL}(n,\mathbb{H})$. This decomposition is given by Salamon [S1, §4], along with more details concerning the nature of the $\mathrm{GL}(n,\mathbb{H})$ representations L_j^k.

If the $\mathrm{Sp}(1)\mathrm{GL}(n,\mathbb{H})$-structure on M reduces to an $\mathrm{Sp}(1)\mathrm{Sp}(n)$-structure, $\Lambda^k T^*M$ can be further decomposed into representations of the compact group $\mathrm{Sp}(1)\mathrm{Sp}(n)$. This refinement is performed in detail by Swann [Sw], and used to demonstrate that if $\dim M \geq 8$, the vanishing condition $\nabla\Omega = 0$ implies that M is quaternionic Kähler for any torsion-free connection ∇ preserving the $\mathrm{Sp}(1)\mathrm{Sp}(n)$-structure.

If we symmetrise completely on V_1 in Equation (7) to obtain V_k, we must antisymmetrise completely on E. Salamon therefore defines the irreducible subspace

$$A^k \cong V_k \otimes \Lambda^k E. \tag{8}$$

The bundle A^k can be described using the decomposition into types for the local almost complex structures on M as follows [S1, Proposition 4.2]: [3]

$$A^k = \sum_{I \in S^2} \Lambda_I^{k,0} M. \tag{9}$$

[3] This is because every $\mathrm{Sp}(1)$-representation V_n is generated by its highest weight spaces taken with respect to all the different linear combinations of I, J and K.

Letting p denote the natural projection $p : \Lambda^k T^*M \to A^k$ and setting $D = p \circ d$, Salamon defines a sequence of differential operators

$$0 \longrightarrow C^\infty(A^0) \xrightarrow{D=d} C^\infty(A^1 = T^*M) \xrightarrow{D} C^\infty(A^2) \xrightarrow{D} \dots \xrightarrow{D} C^\infty(A^{2n}) \longrightarrow 0. \quad (10)$$

This is accomplished using only the fact that M has an $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure; such a manifold is called 'almost quaternionic'. The following theorem of Salamon relates the integrability of such a structure with the sequence of operators in (10):

Theorem 3.2 *[S1, Theorem 4.1] An almost quaternionic manifold is quaternionic if and only if (10) is a complex.*

This theorem is analogous to the familiar result in complex geometry that an almost complex structure on a manifold is integrable if and only if $\overline{\partial}^2 = 0$.

4 Construction of the Double Complex

In this, the most important section of this paper, we construct the new double complex on a quaternionic manifold M by decomposing the action of $\mathrm{Sp}(1)$ on $\Lambda^k T^*M$ inherited from the $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure. The top row of this double complex is the complex (10) discovered by Salamon.

Let M^{4n} be a quaternionic manifold. Following Salamon [S1, §1] we can define (at least locally) vector bundles $\mathbf{V_1}$ and $\mathbf{E}$ associated to the basic complex representations of $\mathrm{Sp}(1)$ and $\mathrm{GL}(n, \mathbb{H})$ respectively, so that $T_x^*M \cong (\mathbf{V_1})_x \otimes \mathbf{E_x} \cong V_1 \otimes E$ as an $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-representation for all $x \in M$. Suppose we consider just the action of the $\mathrm{Sp}(1)$-factor. Then the (complexified) cotangent space effectively takes the form $V_1 \otimes \mathbb{C}^{2n} \cong 2nV_1$. Whilst the bundles $\mathbf{V_1}$ and $\mathbf{E}$ might be neither globally nor uniquely defined, the Casimir operator $I^2 + J^2 + K^2$ is invariant. It follows that, though the $\mathrm{Sp}(1)$-action on a k-form α might be subject to choice, its spectrum under the Casimir action, and hence its decomposition into $\mathrm{Sp}(1)$-representations of different weights, is uniquely and globally defined by the $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-structure. Thus the irreducible decomposition of the $\mathrm{Sp}(1)$-action on $\Lambda^k T_x^*M$ is given by the irreducible decomposition of the representation $\Lambda^k(2nV_1)$.

To work out the irreducible decomposition of this representation we compute the weight space decomposition of $\Lambda^k(2nV_1)$ from that of $2nV_1$. [4] With respect to the action of a particular subgroup $\mathrm{U}(1) \subset \mathrm{Sp}(1)$, the representation $2nV_1$ has weights $+1$ and -1, each occuring with multiplicity $2n$. The weights of $\Lambda^k(2nV_1)$ are the k-wise distinct sums of these. Each weight r in $\Lambda^k(2nV_1)$ must therefore be a sum of p occurences of the weight '+1' and $p - r$ occurences of the weight '−1', where $2p - r = k$ and $0 \leq p \leq k$ (from which it follows immediately that $-k \leq r \leq k$ and $r \equiv k \bmod 2$). The number of ways to choose the p '+1' weights is $\binom{2n}{p}$, and the number of ways to choose the $(p - r)$ '−1' weights is $\binom{2n}{p-r}$, so the multiplicity of the weight r in the representation $\Lambda^k(2nV_1)$ is

$$\mathrm{Mult}(r) = \binom{2n}{\frac{k+r}{2}}\binom{2n}{\frac{k-r}{2}}.$$

[4] This process for calculating the weights of tensor, symmetric and exterior powers is a standard technique in representation theory — see for example [FH, §11.2].

For $r \geq 0$, consider the difference $\text{Mult}(r) - \text{Mult}(r+2)$. This is the number of weight spaces of weight r which do not have any corresponding weight space of weight $r+2$. Each such weight space must therefore be the highest weight space in an irreducible subrepresentation $V_r \subseteq \Lambda^k T^*M$, from which it follows that the number of irreducibles V_r in $\Lambda^k(2nV_1)$ is equal to $\text{Mult}(r) - \text{Mult}(r+2)$. This demonstrates the following proposition:

Proposition 4.1 *Let M^{4n} be a hypercomplex manifold. The decomposition into irreducibles of the induced representation of* $\text{Sp}(1)$ *on $\Lambda^k T^*M$ is*

$$\Lambda^k T^*M \cong \bigoplus_{r=0}^{k} \left[\binom{2n}{\frac{k+r}{2}} \binom{2n}{\frac{k-r}{2}} - \binom{2n}{\frac{k+r+2}{2}} \binom{2n}{\frac{k-r-2}{2}} \right] V_r,$$

where $r \equiv k \bmod 2$.

We will not always write the condition $r \equiv k \bmod 2$, assuming that $\binom{p}{q} = 0$ if $q \notin \mathbb{Z}$.

Definition 4.2 Let M^{4n} be a quaternionic manifold. Define $E_{k,r}$ to be the vector subbundle of $\Lambda^k T^*M$ consisting of $\text{Sp}(1)$-representations with highest weight r. Define the coefficient

$$\epsilon^n_{k,r} = \binom{2n}{\frac{k+r}{2}} \binom{2n}{\frac{k-r}{2}} - \binom{2n}{\frac{k+r+2}{2}} \binom{2n}{\frac{k-r-2}{2}},$$

so that (neglecting the $\text{GL}(n, \mathbb{H})$-action) we have $E_{k,r} \cong \epsilon^n_{k,r} V_r$.

With this notation Proposition 4.1 is written

$$\Lambda^k T^*M \cong \bigoplus_{r=0}^{k} \epsilon^n_{k,r} V_r \cong \bigoplus_{r=0}^{k} E_{k,r}.$$

Our most important result is that this decomposition gives rise to a double complex of differential forms and operators on a quaternionic manifold.

Theorem 4.3 *The exterior derivative d maps $C^\infty(M, E_{k,r})$ to $C^\infty(M, E_{k+1,r+1} \oplus E_{k+1,r-1})$.*

Proof. Let ∇ be a torsion-free linear connection on M preserving the quaternionic structure. Then $\nabla : C^\infty(M, E_{k,r}) \to C^\infty(M, E_{k,r} \otimes T^*M)$. As $\text{Sp}(1)$-representations, this is

$$\nabla : C^\infty(M, \epsilon^n_{k,r} V_r) \to C^\infty(M, \epsilon^n_{k,r} V_r \otimes 2nV_1).$$

Using the Clebsch-Gordon formula we have $\epsilon^n_{k,r} V_r \otimes 2nV_1 \cong 2n\epsilon^n_{k,r}(V_{r+1} \oplus V_{r-1})$. Thus the image of $E_{k,r}$ under ∇ is contained in the V_{r+1} and V_{r-1} summands of $\Lambda^k T^*M \otimes T^*M$. Since ∇ is torsion-free, $d = \wedge \circ \nabla$, so d maps (sections of) $E_{k,r}$ to the V_{r+1} and V_{r-1} summands of $\Lambda^{k+1} T^*M$. Thus $d : C^\infty(M, E_{k,r}) \to C^\infty(M, E_{k+1,r+1} \oplus E_{k+1,r-1})$. ∎

This allows us to split the exterior differential d into two 'quaternionic Dolbeault operators'.

Definition 4.4 Let $\pi_{k,r}$ be the natural projection from $\Lambda^k T^*M$ onto $E_{k,r}$. Define the operators

$$\begin{array}{lcl} \mathcal{D} : C^\infty(E_{k,r}) \to C^\infty(E_{k+1,r+1}) & \quad\text{and}\quad & \overline{\mathcal{D}} : C^\infty(E_{k,r}) \to C^\infty(E_{k+1,r-1}) \\ \mathcal{D} = \pi_{k+1,r+1} \circ d & & \overline{\mathcal{D}} = \pi_{k+1,r-1} \circ d. \end{array} \tag{11}$$

Theorem 4.3 is equivalent to the following:

Proposition 4.5 *On a quaternionic manifold M, we have $d = \mathcal{D} + \overline{\mathcal{D}}$, and so*

$$\mathcal{D}^2 = \mathcal{D}\overline{\mathcal{D}} + \overline{\mathcal{D}}\mathcal{D} = \overline{\mathcal{D}}^2 = 0.$$

Proof. The first equation is equivalent to Theorem 4.3. The rest follows immediately from decomposing the equation $d^2 = 0$. ∎

Figure 3: The Quaternionic Double Complex

$$\begin{tikzcd}
 & & & C^\infty(E_{3,3}) \arrow[dr, "\overline{\mathcal{D}}"] & \ldots\ldots \textit{etc.} \\
 & & C^\infty(E_{2,2}) \arrow[ur, "\mathcal{D}"] \arrow[dr, "\overline{\mathcal{D}}"] & & {} \\
 & C^\infty(E_{1,1} = T^*M) \arrow[ur, "\mathcal{D}"] \arrow[dr, "\overline{\mathcal{D}}"] & & C^\infty(E_{3,1}) \arrow[ur, "\mathcal{D}"] \arrow[dr, "\overline{\mathcal{D}}"] & \ldots\ldots \textit{etc.} \\
C^\infty(E_{0,0} = \mathbb{R}) \arrow[ur, "\mathcal{D}"] & & C^\infty(E_{2,0}) \arrow[ur, "\mathcal{D}"] & & {}
\end{tikzcd}$$

Here is our quaternionic analogue of the Dolbeault complex. There are strong similarities between this and the real Dolbeault complex (Figure 3.1). Again, instead of a diamond as in the Dolbeault complex, the quaternionic version only extends upwards to form an isosceles triangle. This is essentially because there is one irreducible U(1)-representation for each integer, whereas there is one irreducible Sp(1)-representation only for each nonnegative integer.

By definition, the bundle $E_{k,k}$ is the bundle A^k of (8) — they are both the subbundle of $\Lambda^k T^*M$ which includes all Sp(1)-representations of the form V_k. Thus the leading edge of the double complex

$$0 \longrightarrow C^\infty(E_{0,0}) \xrightarrow{\mathcal{D}} C^\infty(E_{1,1}) \xrightarrow{\mathcal{D}} C^\infty(E_{2,2}) \xrightarrow{\mathcal{D}} \ldots \xrightarrow{\mathcal{D}} C^\infty(E_{2n,2n}) \xrightarrow{\mathcal{D}} 0$$

is precisely the complex (10) discovered by Salamon.

Example 4.6 Four Dimensions

This double complex is already very well-known and understood in four dimensions. Here there is a splitting only in the middle dimension, $\Lambda^2 T^*M \cong V_2 \oplus 3V_0$. Let I, J and K be local almost complex structures at $x \in M$, and let $e^0 \in T^*_x M$. Let $e^1 = I(e^0)$, $e^2 = J(e^0)$ and $e^3 = K(e^0)$. In this way we obtain a basis $\{e^0, \ldots, e^3\}$ for $T^*_x M \cong \mathbb{H}$. Using the notation $e^{ijk\ldots} = e^i \wedge e^j \wedge e^k \wedge \ldots etc.$, define the 2-forms

$$\omega_1^\pm = e^{01} \pm e^{23}, \quad \omega_2^\pm = e^{02} \pm e^{31}, \quad \omega_3^\pm = e^{03} \pm e^{12}. \tag{12}$$

Then I, J and K all act trivially on the ω_j^-, so $E_{2,0} = \langle \omega_1^-, \omega_2^-, \omega_3^- \rangle$. The action of $\mathfrak{sp}(1)$ on the ω_j^+ is given by the multiplication table

$$\begin{array}{rclrclrcl} I(\omega_1^+) & = & 0 & I(\omega_2^+) & = & 2\omega_3^+ & I(\omega_3^+) & = & -2\omega_3^+ \\ J(\omega_1^+) & = & -2\omega_3^+ & J(\omega_2^+) & = & 0 & J(\omega_3^+) & = & 2\omega_1^+ \\ K(\omega_1^+) & = & 2\omega_2^+ & K(\omega_2^+) & = & -2\omega_1^+ & K(\omega_3^+) & = & 0. \end{array} \tag{13}$$

These are the relations of the irreducible $\mathfrak{sp}(1)$-representation V_2, and we see that $E_{2,2} = \langle \omega_1^+, \omega_2^+, \omega_3^+ \rangle$.

These bundles will be familiar to most readers: $E_{2,2}$ is the bundle of *self-dual* 2-forms Λ^2_+ and $E_{2,0}$ is the bundle of *anti-self-dual* 2-forms Λ^2_-. The celebrated splitting $\Lambda^2 T^*M \cong \Lambda^2_+ \oplus \Lambda^2_-$ is an invariant of the conformal class of any Riemannian 4-manifold, and $I^2 + J^2 + K^2 = -4(* + 1)$, where $* : \Lambda^k T^*M \to \Lambda^{4-k} T^*M$ is the Hodge star map.

This also serves to explain the special definition that a 4-manifold is said to be quaternionic if it is self-dual and conformal. The relationship between quaternionic, almost complex and Riemannian structures in four dimensions is described in detail in [S2, Chapter 7].

Because there is no suitable quaternionic version of holomorphic coordinates, there is no 'nice' co-ordinate expression for a typical section of $C^\infty(E_{k,r})$. In order to determine the decomposition of a differential form, the simplest way the author has found is to use the Casimir operator $\mathcal{C} = I^2 + J^2 + K^2$. Consider a k-form α. Then $\alpha \in E_{k,r}$ if and only if $(I^2 + J^2 + K^2)(\alpha) = -r(r+2)\alpha$. This mechanism also allows us to work out expressions for $\mathcal{D}$ and $\overline{\mathcal{D}}$ acting on α.

Lemma 4.7 *Let $\alpha \in C^\infty(E_{k,r})$. Then*

$$\mathcal{D}\alpha = -\frac{1}{4}\left((r-1) + \frac{1}{r+1}(I^2 + J^2 + K^2)\right) d\alpha$$

and

$$\overline{\mathcal{D}}\alpha = \frac{1}{4}\left((r+3) + \frac{1}{r+1}(I^2 + J^2 + K^2)\right) d\alpha.$$

Proof. We have $d\alpha = \mathcal{D}\alpha + \overline{\mathcal{D}}\alpha$, where $\mathcal{D}\alpha \in E_{k+1,r+1}$ and $\overline{\mathcal{D}}\alpha \in E_{k+1,r-1}$. Applying the Casimir operator gives

$$(I^2 + J^2 + K^2)(d\alpha) = -(r+1)(r+3)\mathcal{D}\alpha - (r+1)(r-1)\overline{\mathcal{D}}\alpha.$$

Rearranging these equations gives $\mathcal{D}\alpha$ and $\overline{\mathcal{D}}\alpha$. ∎

Note that our decomposition is of real- as well as complex- valued forms; the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ map real forms to real forms.

Writing $\mathcal{D}_{k,r}$ for the particular map $\mathcal{D} : C^\infty(E_{k,r}) \to C^\infty(E_{k+1,r+1})$, we define the quaternionic cohomology groups

$$H^{k,r}_{\mathcal{D}}(M) = \frac{\mathrm{Ker}(\mathcal{D}_{k,r})}{\mathrm{Im}(\mathcal{D}_{k-1,r-1})}. \tag{14}$$

5 Ellipticity and the Double Complex

In this section we shall determine where our double complex is elliptic and where it is not. Its properties are extremely like those of the real Dolbeault complex studied earlier: the quaternionic double complex is elliptic everywhere except on the bottom row. Though this is much more difficult to prove for the quaternionic double complex, the fundamental reason is the same as for the real Dolbeault complex: it is the isosceles triangle as opposed to diamond shape which causes ellipticity to fail for the bottom row, because $d = \mathcal{D}$ on $E_{2k,0}$ and the projection from $d(C^\infty(E_{2k,0}))$ to $C^\infty(E_{2k+1,1})$ is the identity.

Here is the main result of this section:

Theorem 5.1 *For $2k \geq 4$, the complex*

$$0 \to E_{2k,0} \xrightarrow{\mathcal{D}} E_{2k+1,1} \xrightarrow{\mathcal{D}} E_{2k+2,2} \xrightarrow{\mathcal{D}} \ldots \xrightarrow{\mathcal{D}} E_{2n+k,2n-k} \xrightarrow{\mathcal{D}} 0$$

is elliptic everywhere except at $E_{2k,0}$ and $E_{2k+1,1}$, where it is not elliptic.

For $k = 1$ the complex is elliptic everywhere except at $E_{3,1}$, where it is not elliptic.

For $k = 0$ the complex is elliptic everywhere.

The rest of this section provides a proof of this theorem.

On a complex manifold M^{2n} with holomorphic coordinates z^j, the exterior forms $dz^{a_1} \wedge \ldots \wedge dz^{a_p} \wedge d\bar{z}^{b_1} \wedge \ldots \wedge d\bar{z}^{b_q}$ span $\Lambda^{p,q}$. This allows us to decompose any form $\omega \in \Lambda^{p,q}$, making it much easier to write down the kernels and images of maps which involve exterior multiplication. On a quaternionic manifold M^{4n} there is unfortunately no easy way to write down a local frame for the bundle $E_{k,r}$, because there is no quaternionic version of 'holomorphic coordinates'. However, we can decompose $E_{k,r}$ just enough to enable us to prove Theorem 5.1.

A principal observation is that since ellipticity is a local property, we can work on $\mathbb{H}^n$ without loss of generality. Secondly, since $\mathrm{GL}(n, \mathbb{H})$ acts transitively on $\mathbb{H}^n \setminus \{0\}$, if the symbol sequence $\ldots \xrightarrow{\sigma_{e^0}} E_{k,r} \xrightarrow{\sigma_{e^0}} E_{k+1,r+1} \xrightarrow{\sigma_{e^0}} \ldots$ is exact for any nonzero $e^0 \in T^*\mathbb{H}^n$ then it is exact for all nonzero $\xi \in T^*\mathbb{H}^n$. To prove Theorem 5.1, we choose one such e^0 and analyse the spaces $E_{k,r}$ accordingly.

5.1 Decomposition of the Spaces $E_{k,r}$

Let $e^0 \in T^*_x\mathbb{H}^n \cong \mathbb{H}^n$ and let (I, J, K) be the standard hypercomplex structure on $\mathbb{H}^n$. As in Example 4.6, define $e^1 = I(e^0)$, $e^2 = J(e^0)$ and $e^3 = K(e^0)$, so that $\langle e^0, \ldots, e^3\rangle \cong \mathbb{H}$. In this way we single out a particular copy of $\mathbb{H}$ which we call $\mathbb{H}_0$, obtaining a (nonnatural)

splitting $T_x^*\mathbb{H}^n \cong \mathbb{H}^{n-1} \oplus \mathbb{H}_0$ which is preserved by action of the hypercomplex structure. This induces the decomposition $\Lambda^k\mathbb{H}^n \cong \bigoplus_{l=0}^4 \Lambda^{k-l}\mathbb{H}^{n-1} \otimes \Lambda^l\mathbb{H}_0$, which decomposes each $E_{k,r} \subset \Lambda^k\mathbb{H}^n$ according to how many differentials in the $\mathbb{H}_0$-direction are present.

Definition 5.2 Define the space $E_{k,r}^l$ to be the subspace of $E_{k,r}$ consisting of exterior forms with precisely l differentials in the $\mathbb{H}_0$-direction, *i.e.*

$$E_{k,r}^l \equiv E_{k,r} \cap (\Lambda^{k-l}\mathbb{H}^{n-1} \otimes \Lambda^l\mathbb{H}_0).$$

Then $E_{k,r}^l$ is preserved by the induced action of the hypercomplex structure on $\Lambda^k\mathbb{H}^n$. Thus we obtain an invariant decomposition $E_{k,r} = E_{k,r}^0 \oplus E_{k,r}^1 \oplus E_{k,r}^2 \oplus E_{k,r}^3 \oplus E_{k,r}^4$. Note that we can identify $E_{k,r}^0$ on $\mathbb{H}^n$ with $E_{k,r}$ on $\mathbb{H}^{n-1}$.

(Throughout the rest of this section, juxtaposition of exterior forms will denote exterior multiplication, for example αe^{ij} means $\alpha \wedge e^{ij}$.)

We can decompose these summands still further. Consider, for example, the bundle $E_{k,r}^1$. An exterior form $\alpha \in E_{k,r}^1$ is of the form $\alpha_0 e^0 + \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3$, where $\alpha_j \in \Lambda^{k-1}\mathbb{H}^{n-1}$. Thus α is an element of $\Lambda^{k-1}\mathbb{H}^{n-1} \otimes 2V_1$, since $\mathbb{H}_0 \cong 2V_1$ as an $\mathfrak{sp}(1)$-representation. Since α is in a copy of the representation V_r, it follows from the isomorphism $V_r \otimes V_1 \cong V_{r+1} \oplus V_{r-1}$ that the α_j must be in a combination of V_{r+1} and V_{r-1} representations, *i.e.* $\alpha_j \in E_{k-1,r+1}^0 \oplus E_{k-1,r-1}^0$. We write

$$\alpha = \alpha^+ + \alpha^- = (\alpha_0^+ + \alpha_0^-)e^0 + (\alpha_1^+ + \alpha_1^-)e^1 + (\alpha_2^+ + \alpha_2^-)e^2 + (\alpha_3^+ + \alpha_3^-)e^3,$$

where $\alpha_j^+ \in E_{k-1,r+1}^0$ and $\alpha_j^- \in E_{k-1,r-1}^0$.

The following Lemma allows us to consider α^+ and α^- separately.

Lemma 5.3 *If $\alpha = \alpha^+ + \alpha^- \in E_{k,r}^1$ then both α^+ and α^- must be in $E_{k,r}^1$.*

Proof. In terms of representations, the situation is of the form

$$(pV_{r+1} \oplus qV_{r-1}) \otimes 2V_1 \cong 2p(V_{r+2} \oplus V_r) \oplus 2q(V_r \oplus V_{r-2}),$$

where $\alpha^+ \in pV_{r+1}$ and $\alpha^- \in qV_{r-1}$. For α to be in the representation $2(p+q)V_r$, its components in the representations $2pV_{r+2}$ and $2qV_{r-2}$ must both vanish separately. The component in $2pV_{r+2}$ comes entirely from α^+, so for this to vanish we must have $\alpha^+ \in 2(p+q)V_r$ independently of α^-. Similarly, for the component in $2qV_{r-2}$ to vanish, we must have $\alpha^- \in 2(p+q)V_r$. ∎

Thus we decompose the space $E_{k,r}^1$ into two summands, one coming from $E_{k-1,r-1}^0 \otimes 2V_1$ and one from $E_{k-1,r+1}^0 \otimes 2V_1$. We extend this decomposition to the cases $l = 0, 2, 3, 4$, defining the following notation.

Definition 5.4 Define the bundle $E_{k,r}^{l,m}$ to be the subbundle of $E_{k,r}^l$ arising from V_m-type representations in $\Lambda^{k-l}\mathbb{H}^{n-1}$. In other words,

$$E_{k,r}^{l,m} \equiv (E_{k-l,m}^0 \otimes \Lambda^l\mathbb{H}_0) \cap E_{k,r}.$$

To recapitulate: for the space $E_{k,r}^{l,m}$, the bottom left index k refers to the exterior power of the form $\alpha \in \Lambda^k \mathbb{H}^n$; the bottom right index r refers to the irreducible Sp(1)-representation in which α lies; the top left index l refers to the number of differentials in the $\mathbb{H}_0$-direction and the top right index m refers to the irreducible Sp(1)-representation of the contributions from $\Lambda^{k-a}\mathbb{H}^{n-1}$ *before* wedging with forms in the $\mathbb{H}_0$-direction. This may appear slightly fiddly: it becomes rather simpler when we consider the specific splittings which Definition 5.4 allows us to write down.

Lemma 5.5 *Let $E_{k,r}^{l,m}$ be as above. We have the following decompositions:*

$$E_{k,r}^0 = E_{k,r}^{0,r} \qquad E_{k,r}^1 = E_{k,r}^{1,r+1} \oplus E_{k,r}^{1,r-1} \qquad E_{k,r}^2 = E_{k,r}^{2,r+2} \oplus E_{k,r}^{2,r} \oplus E_{k,r}^{2,r-2}$$

$$E_{k,r}^3 = E_{k,r}^{3,r+1} \oplus E_{k,r}^{3,r-1} \qquad \textit{and} \qquad E_{k,r}^4 = E_{k,r}^{4,r}.$$

Proof. The first isomorphism is trivial, as is the last (since the hypercomplex structure acts trivially on $\Lambda^4\mathbb{H}_0$). The second isomorphism is Lemma 5.3, and the fourth follows in exactly the same way since $\Lambda^3\mathbb{H}_0 \cong 2V_1$ also. The middle isomorphism follows a similar argument. ∎

Recall the self-dual forms and anti-self-dual forms in Example 4.6. The bundle $E_{k,r}^{2,r}$ splits according to whether its contribution from $\Lambda^2\mathbb{H}_0$ is self-dual or anti-self-dual. We will call these summands $E_{k,r}^{2,r+}$ and $E_{k,r}^{2,r-}$ respectively, so $E_{k,r}^{2,r} = E_{k,r}^{2,r+} \oplus E_{k,r}^{2,r-}$.

5.2 Lie in conditions

We have analysed the bundle $E_{k,r}$ into a number of different subbundles. We now determine when a particular exterior form lies in one of these subbundles. Consider a form $\alpha = \alpha_1 e^{s_1 \ldots s_a} + \alpha_2 e^{t_1 \ldots t_a} + \ldots$ *etc.* where $\alpha_j \in E_{k-a,b}^0$. For α to lie in one of the spaces $E_{k,r}^{a,b}$ the α_j will usually have to satisfy some simultaneous equations. Since these are the conditions for a form to lie in a particular Lie algebra representation, we will refer to such equations as 'Lie In Conditions'.

To begin with, we mention three trivial lie in conditions. Let $\alpha \in E_{k,r}^0$. That $\alpha \in E_{k,r}^{0,r}$ is obvious, as is $\alpha e^{0123} \in E_{k,r}^{4,r}$, since wedging with e^{0123} has no effect on the $\mathfrak{sp}(1)$-action. Likewise, the $\mathfrak{sp}(1)$-action on the anti-self-dual 2-forms $\omega_1^- = e^{01} - e^{23}$, $\omega_2^- = e^{02} - e^{31}$ and $\omega_3^- = e^{03} - e^{12}$ is trivial, so $\alpha\omega_j^- \in E_{k,r}^{2,r-}$ for all $j = 1, 2, 3$.

This leaves the following three situations: those arising from taking exterior products with 1-forms, 3-forms and the self-dual 2-forms ω_j^+. As usual when we want to know which representation an exterior form is in, we apply the Casimir operator.

5.2.1 The cases $l = 1$ and $l = 3$

Let $\alpha_j \in E_{k,r}^0$. Then $\alpha = \alpha_0 e^0 + \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3 \in E_{k+1,r+1}^{1,r} \oplus E_{k+1,r-1}^{1,r}$, and α is entirely in $E_{k+1,r+1}^{1,r}$ if and only if $(I^2 + J^2 + K^2)\alpha = -(r+1)(r+3)\alpha$.

By the usual (Leibniz) rule for a Lie algebra action on a tensor product, we have that $I^2(\alpha_j e^j) = I^2(\alpha_j)e^j + 2I(\alpha_j)I(e^j) + \alpha_j I^2(e^j)$, *etc.* Thus

$$
\begin{aligned}
(I^2+J^2+K^2)\alpha &= \sum_{j=0}^{3}\Big[(I^2+J^2+K^2)(\alpha_j)e^j + \alpha_j(I^2+J^2+K^2)(e^j) \;+ \\
&\qquad + 2\Big(I(\alpha_j)I(e^j)+J(\alpha_j)J(e^j)+K(\alpha_j)K(e^j)\Big)\Big] \\
&= -r(r+2)\alpha - 3\alpha + 2\sum_{j=0}^{3}\Big(I(\alpha_j)I(e^j)+J(\alpha_j)J(e^j)+K(\alpha_j)K(e^j)\Big) \\
&= (-r^2-2r-3)\alpha + 2\,\Big(I(\alpha_0)e^1 - I(\alpha_1)e^0 + I(\alpha_2)e^3 - I(\alpha_3)e^2 + \\
&\qquad +J(\alpha_0)e^2 - J(\alpha_1)e^3 - J(\alpha_2)e^0 + J(\alpha_3)e^1 + \\
&\qquad +K(\alpha_0)e^3 + K(\alpha_1)e^2 - K(\alpha_2)e^1 - K(\alpha_3)e^0\Big).
\end{aligned} \tag{15}
$$

For $\alpha \in E^{1,r}_{k+1,r+1}$ we need this to be equal to $-(r+1)(r+3)\alpha$, which is the case if and only if

$$
\begin{aligned}
-r\alpha &= I(\alpha_0)e^1 - I(\alpha_1)e^0 + I(\alpha_2)e^3 - I(\alpha_3)e^2 + J(\alpha_0)e^2 - J(\alpha_1)e^3 - J(\alpha_2)e^0 + J(\alpha_3)e^1 + \\
&\quad + K(\alpha_0)e^3 + K(\alpha_1)e^2 - K(\alpha_2)e^1 - K(\alpha_3)e^0.
\end{aligned}
$$

Since the α_j have no e^j-factors and the action of I, J and K preserves this property, this equation can only be satisfied if it holds for each of the e^j-components separately. It follows that $\alpha \in E^{1,r}_{k+1,r+1}$ if and only if α_0, α_1, α_2 and α_3 satisfy the following lie in conditions: [5]

$$
\begin{aligned}
r\alpha_0 - I(\alpha_1) - J(\alpha_2) - K(\alpha_3) &= 0 \\
r\alpha_1 + I(\alpha_0) + J(\alpha_3) - K(\alpha_2) &= 0 \\
r\alpha_2 - I(\alpha_3) + J(\alpha_0) + K(\alpha_1) &= 0 \\
r\alpha_3 + I(\alpha_2) - J(\alpha_1) + K(\alpha_0) &= 0.
\end{aligned} \tag{16}
$$

Suppose instead that $\alpha \in E^{1,r}_{k+1,r-1}$. Then $(I^2+J^2+K^2)\alpha = -(r-1)(r+1)\alpha$. Putting this alternative into Equation (15) gives the result that $\alpha \in E^{1,r}_{k+1,r-1}$ if and only if

$$
\begin{aligned}
(r+2)\alpha_0 + I(\alpha_1) + J(\alpha_2) + K(\alpha_3) &= 0 \\
(r+2)\alpha_1 - I(\alpha_0) - J(\alpha_3) + K(\alpha_2) &= 0 \\
(r+2)\alpha_2 + I(\alpha_3) - J(\alpha_0) - K(\alpha_1) &= 0 \\
(r+2)\alpha_3 - I(\alpha_2) + J(\alpha_1) - K(\alpha_0) &= 0.
\end{aligned} \tag{17}
$$

Consider now $\alpha = \alpha_0 e^{123} + \alpha_1 e^{032} + \alpha_2 e^{013} + \alpha_3 e^{021} \in E^{3,r}_{k+3,r+1} \oplus E^{3,r}_{k+3,r-1}$. Since $\Lambda^3\mathbb{H}_0 \cong \mathbb{H}_0$, the lie in conditions are exactly the same: for α to be in $E^{3,r}_{k+3,r+1}$ we need the α_j to satisfy Equations (16), and for α to be in $E^{3,r}_{k+3,r-1}$ we need the α_j to satisfy Equations (17).

[5] Our interest in these conditions arises from a consideration of exterior forms, but the equations describe $\mathfrak{sp}(1)$-representations in general: they are the conditions that $\alpha \in V_r \otimes V_1$ must satisfy to be in the V_{r+1} subspace of $V_{r+1} \oplus V_{r-1} \cong V_r \otimes V_1$. The other lie in conditions have similar interpretations.

5.2.2 The case $l = 2$

We have already noted that wedging a form $\beta \in E^0_{k,r}$ with an anti-self-dual 2-form ω_j^- has no effect on the $\mathfrak{sp}(1)$-action, so $\beta\omega_j^- \in E^{2,r-}_{k+2,r}$. Thus we only have to consider the effect of wedging with the self-dual 2-forms $\langle \omega_1^+, \omega_2^+, \omega_3^+ \rangle \cong V_2 \subset \Lambda^2\mathbb{H}_0$. By the Clebsch-Gordon formula, the decomposition takes the form $V_r \otimes V_2 \cong V_{r+2} \oplus V_r \oplus V_{r-2}$. Thus for $\beta = \beta_1\omega_1^+ + \beta_2\omega_2^+ + \beta_3\omega_3^+$ we want to establish the lie in conditions for β to be in $E^{2,r}_{k+2,r+2}$, $E^{2,r+}_{k+2,r}$ and $E^{2,r}_{k+2,r-2}$.

We calculate these lie in conditions in a similar fashion to the previous cases, by considering the action of the Casimir operator $I^2 + J^2 + K^2$ on β and using the multiplication table (13). The following lie in conditions are then easy to deduce:

$$\beta \in E^{2,r}_{k+2,r+2} \Longleftrightarrow \begin{cases} (r+4)\beta_1 &= J(\beta_3) - K(\beta_2) \\ (r+4)\beta_2 &= K(\beta_1) - I(\beta_3) \\ (r+4)\beta_3 &= I(\beta_2) - J(\beta_1). \end{cases} \tag{18}$$

$$\beta \in E^{2,r+}_{k+2,r} \Longleftrightarrow \begin{cases} 2\beta_1 &= J(\beta_3) - K(\beta_2) \\ 2\beta_2 &= K(\beta_1) - I(\beta_3) \\ 2\beta_3 &= I(\beta_2) - J(\beta_1). \end{cases} \tag{19}$$

$$\beta \in E^{2,r}_{k+2,r-2} \Longleftrightarrow \begin{cases} (2-r)\beta_1 &= J(\beta_3) - K(\beta_2) \\ (2-r)\beta_2 &= K(\beta_1) - I(\beta_3) \\ (2-r)\beta_3 &= I(\beta_2) - J(\beta_1). \end{cases} \tag{20}$$

Equation (19) is particularly interesting. Since this equation singles out the V_r-representation in the direct sum $V_{r+2} \oplus V_r \oplus V_{r+2}$, it must have $\dim V_r = r+1$ linearly independent solutions. Let $\beta_0 \in V_r$ and let $\beta_1 = I(\beta_0)$, $\beta_2 = J(\beta_0)$, $\beta_3 = K(\beta_0)$. Using the Lie algebra relations $2I = [J, K] = JK - KJ$, it is easy to see that β_1, β_2 and β_3 satisfy Equation 19. Moreover, there are $r+1$ linearly independent solutions of this form (for $r \neq 0$). We conclude that *all* the solutions of Equation (19) take the form $\beta_1 = I(\beta_0)$, $\beta_2 = J(\beta_0)$, $\beta_3 = K(\beta_0)$.

5.3 The Symbol Sequence and Proof of Theorem 5.1

We now describe the principal symbol of $\mathcal{D}$, and examine its behaviour in the context of the decompositions of Definition 5.2 and Lemma 5.5. This leads to a proof of Theorem 5.1. First we obtain the principal symbol from the formula for $\mathcal{D}$ in Lemma 4.7 by replacing $d\alpha$ with αe^0.

Proposition 5.6 *Let $x \in \mathbb{H}^n$, $e^0 \in T^*_x\mathbb{H}^n$ and $\alpha \in E_{k,r}$. The principal symbol mapping $\sigma_{\mathcal{D}}(x, e^0) : E_{k,r} \to E_{k+1,r+1}$ is given by*

$$\sigma_{\mathcal{D}}(x, e^0)(\alpha) = \frac{1}{2(r+1)} \left((r+2)\alpha e^0 - I(\alpha)e^1 - J(\alpha)e^2 - K(\alpha)e^3 \right).$$

Proof. Replacing $d\alpha$ with αe^0 in the formula for $\mathcal{D}$ obtained in Lemma 4.7, we have

$$\begin{aligned}
\sigma_{\mathcal{D}}(x,e^0)(\alpha) &= -\frac{1}{4}\Big((r-1)+\frac{1}{r+1}(I^2+J^2+K^2)\Big)\alpha e^0 \\
&= \frac{-1}{4(r+1)}\Big[\,(\,(r-1)(r+1)-r(r+2)-3\,)\,\alpha e^0 \\
&\qquad +2\left(I(\alpha)e^1+J(\alpha)e^2+K(\alpha)e^3\right)\Big] \\
&= \frac{1}{2(r+1)}\Big((r+2)\alpha e^0-I(\alpha)e^1-J(\alpha)e^2-K(\alpha)e^3\Big),
\end{aligned}$$

as required. ∎

Corollary 5.7 *The principal symbol $\sigma_{\mathcal{D}}(x,e^0)$ maps the space $E^{l,m}_{k,r}$ to the space $E^{l+1,m}_{k+1,r+1}$.*

Proof. We already know that $\sigma_{\mathcal{D}}: E_{k,r}\to E_{k+1,r+1}$, by definition. Using Lemma 5.6, we see that $\sigma_{\mathcal{D}}(x,e^0)$ increases the number of differentials in the $\mathbb{H}_0$-direction by one, so the index l increases by one. The only action in the other directions is the $\mathfrak{sp}(1)$-action, which preserves the irreducible decomposition of the contribution from $\Lambda^{k-a}\mathbb{H}^{n-1}$, so the index m remains the same. ∎

To save space we shall use σ as an abbreviation for $\sigma_{\mathcal{D}}(x,e^0)$ for the rest of this section. The point of all this work on decomposition now becomes apparent. Since $\sigma: E^l_{k,r}\to E^{l+1}_{k+1,r+1}$, we can reduce the indefinitely long symbol sequence

$$\ldots\xrightarrow{\sigma}E_{k-1,r-1}\xrightarrow{\sigma}E_{k,r}\xrightarrow{\sigma}E_{k+1,r+1}\xrightarrow{\sigma}\ldots\ etc.$$

to the 5-space sequence

$$0\xrightarrow{\sigma}E^0_{k-2,r-2}\xrightarrow{\sigma}E^1_{k-1,r-1}\xrightarrow{\sigma}E^2_{k,r}\xrightarrow{\sigma}E^3_{k+1,r+1}\xrightarrow{\sigma}E^4_{k+2,r+2}\xrightarrow{\sigma}0. \tag{21}$$

Using Lemma 5.5 as well, we can analyse this sequence still further according to the different (top right) m-indices, obtaining three short sequences (for $k\geq 2$, $k\equiv r \bmod 2$)

$$\begin{array}{ccccccccccccc}
 & & & & 0 & \to & E^{2,r+2}_{k,r} & \to & E^{3,r+2}_{k+1,r+1} & \to & E^{4,r+2}_{k+2,r+2} & \to & 0 \\
 & & & & & & \oplus & & \oplus & & & & \\
 & & 0 & \to & E^{1,r}_{k-1,r-1} & \to & E^{2,r}_{k,r} & \to & E^{3,r}_{k+1,r+1} & \to & 0 & & \\
 & & & & \oplus & & \oplus & & & & & & \\
0 & \to & E^{0,r-2}_{k-2,r-2} & \to & E^{1,r-2}_{k-1,r-1} & \to & E^{2,r-2}_{k,r} & \to & 0. & & & &
\end{array} \tag{22}$$

This reduces the problem of determining where the operator $\mathcal{D}$ is elliptic to the problem of determining when these three sequences are exact.

For a sequence $0\to A\to B\to C\to 0$ to be exact, it is necessary that $\dim A-\dim B+\dim C=0$. Given this condition, if the sequence is exact at any two out of A, B and C it is exact at the third. We shall show that for $r\neq 0$ this dimension sum does equal zero.

Lemma 5.8 *For $r>0$, each of the sequences in (22) satisfies the dimension condition above, i.e. the alternating sum of the dimensions vanishes.*

Proof. Let $r > 0$. We calculate the dimensions of the spaces $E^{l,m}_{k,r}$ for $l = 0, \dots, 4$. Recall the notation $E_{k,r} = \epsilon^n_{k,r} V_r$ from Definition 4.2. It is clear that $\dim E^0_{k,r} = (r+1)\epsilon^{n-1}_{k,r}$, since $E^0_{k,r}$ on $\mathbb{H}^n$ is sinply $E_{k,r}$ on $\mathbb{H}^{n-1}$. Thus $\dim E^{0,r-2}_{k-2,r-2} = (r-1)\epsilon^{n-1}_{k-2,r-2}$ and $\dim E^{4,r+2}_{k+2,r+2} = (r+3)\epsilon^{n-1}_{k-2,r+2}$.

The cases $a = 1$ and $a = 3$ are easy to work out since they are of the form $E^0_{k,r} \otimes 2V_1$. For $a = 1$, we have $\dim E^{1,r-2}_{k-1,r-1} = 2r\epsilon^{n-1}_{k-2,r-2}$ and $\dim E^{1,r}_{k-1,r-1} = 2r\epsilon^{n-1}_{k-2,r}$. For $a = 3$, $\dim E^{3,r}_{k+1,r+1} = 2(r+2)\epsilon^{n-1}_{k-2,r}$ and $\dim E^{3,r+2}_{k+1,r+1} = 2(r+2)\epsilon^{n-1}_{k-2,r+2}$.

The case $a = 2$ is slightly more complicated, as we have to take into account exterior products with the self-dual 2-forms V_2 and anti-self-dual 2-forms V_0 in $\Lambda^2\mathbb{H}_0$. The spaces $E^{2,r+2}_{k,r}$ and $E^{2,r-2}_{k,r}$ receive contributions only from the self-dual part V_2, from which we infer that $\dim E^{2,r+2}_{k,r} = (r+1)\epsilon^{n-1}_{k-2,r+2}$ and $\dim E^{2,r-2}_{k,r} = (r+1)\epsilon^{n-1}_{k-2,r-2}$. Finally, the space $E^{2,r+}_{k,r}$ has dimension $(r+1)\epsilon^{n-1}_{k-2,r}$ and the space $E^{2,r-}_{k,r}$ has dimension $3(r+1)\epsilon^{n-1}_{k-2,r}$, giving $E^{2,r}_{k,r}$ a total dimension of $4(r+1)\epsilon^{n-1}_{k-2,r}$.

It is now a simple matter to verify that for the top sequence of (22)

$$\epsilon^{n-1}_{k-2,r+2}(r + 1 - 2(r+2) + r + 3) = 0,$$

for the middle sequence

$$\epsilon^{n-1}_{k-2,r}(2r - 4(r+1) + 2(r+2)) = 0,$$

and for the bottom sequence

$$\epsilon^{n-1}_{k-2,r-2}(r - 1 - 2r + r + 1) = 0.$$

∎

The case $r = 0$ is different. Here the bottom sequence of (22) disappears altogether, the top sequence still being exact. Exactness is lost in the middle sequence. Since the isomorphism $\epsilon^{n-1}_{k-2,0}V_0 \otimes V_2 \cong \epsilon^{n-1}_{k-2,0}V_2$ gives no trivial V_0-representations, there is no space $E^{2,0+}_{k,0}$. Thus $E^{2,0}_{k,0}$ is 'too small' — we are left with a sequence

$$0 \longrightarrow 3\epsilon^{n-1}_{k-2,0}V_0 \longrightarrow 2\epsilon^{n-1}_{k-2,0}V_1 \longrightarrow 0,$$

which cannot be exact. (As there is no space $E^2_{0,0}$, this problem does not arise for the leading edge $0 \to E_{0,0} \to E_{1,1} \to \dots$ *etc.*)

We are finally in a position to prove Theorem 5.1, which now follows from:

Proposition 5.9 *When $r \neq 0$, the three sequences of (22) are exact.*

Proof. Consider first the top sequence $0 \xrightarrow{\sigma} E^{2,r+2}_{k,r} \xrightarrow{\sigma} E^{3,r+2}_{k+1,r+1} \xrightarrow{\sigma} E^{4,r+2}_{k+2,r+2} \longrightarrow 0$. The Clebsch-Gordon formula shows that there are no spaces $E^{3,r+2}_{k+1,r-1}$ or $E^{4,r+2}_{k+2,r}$. Thus $\overline{\mathcal{D}} = 0$ on $E^{2,r+2}_{k,r}$ and $E^{3,r+2}_{k+1,r+1}$, so $\mathcal{D} = d$ for the top sequence. It is easy to check using the relevant lie in conditions that the map $\wedge e^0 : E^{2,r+2}_{k,r} \to E^{3,r+2}_{k+1,r+1}$ is injective and the map $\wedge e^0 : E^{3,r+2}_{k+1,r+1} \to E^{4,r+2}_{k+2,r+2}$ is surjective.

To show exactness at $E^1_{k-1,r-1}$, consider $\alpha = \alpha_0 e^0 + \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3 \in E^1_{k-1,r-1}$. A calculation using Proposition 5.6 shows that

$$\begin{aligned}\sigma(\alpha) &= \frac{1}{2r}\big((r\alpha_1 + I(\alpha_0))e^{10} + (r\alpha_2 + J(\alpha_0))e^{20} + (r\alpha_3 + K(\alpha_0))e^{30} + \\ &\quad + (2\alpha_1 - J(\alpha_3) + K(\alpha_2))e^{32} + (2\alpha_2 - K(\alpha_1) + I(\alpha_3))e^{13} + (2\alpha_3 - I(\alpha_2) + K(\alpha_1))e^{21}\big).\end{aligned} \tag{23}$$

Since the α_i have no e^j-components, $\sigma(\alpha) = 0$ if and only if all these components vanish. This occurs if and only if $\alpha_1 = -\frac{1}{r}I(\alpha_0)$, $\alpha_2 = -\frac{1}{r}J(\alpha_0)$, $\alpha_3 = -\frac{1}{r}K(\alpha_0)$ (since as remarked in Section 5.1 these equations also guarantee that $2\alpha_1 - J(\alpha_3) + K(\alpha_2) = 0$ *etc.*), in which case it is clear that

$$\sigma(\alpha) = 0 \Longleftrightarrow \alpha = \sigma\left(\tfrac{2(r+1)}{r}\alpha_0\right).$$

This shows that the sequence $E^0_{k-2,r-2} \to E^1_{k-1,r-1} \to E^2_{k,r}$ is exact. Restricting to $E^{1,r}_{k-1,r-1}$ and $E^{1,r-2}_{k-1,r-1}$, we see that exactness holds at these spaces in the middle and bottom sequences respectively of (22).

Consider $\alpha \in E^0_{k-2,r-2}$. Then

$$\sigma(\alpha) = \frac{1}{2(r-1)}\left(r\alpha e^0 - I(\alpha)e^1 - J(\alpha)e^2 - K(\alpha)e^3\right).$$

Since these are linearly independent, $\sigma(\alpha) = 0$ if and only if $\alpha = 0$, and $\sigma : E^{0,r-2}_{k-2,r-2} \to E^{1,r-2}_{k-1,r-1}$ is injective. Hence the bottom sequence $0 \longrightarrow E^{0,r-2}_{k-2,r-2} \xrightarrow{\sigma} E^{1,r-2}_{k-1,r-1} \xrightarrow{\sigma} E^{2,r-2}_{k,r} \longrightarrow 0$ is exact.

Finally, we show that the middle sequence $0 \longrightarrow E^{1,r}_{k-1,r-1} \xrightarrow{\sigma} E^{2,r}_{k,r} \xrightarrow{\sigma} E^{3,r}_{k+1,r+1} \longrightarrow 0$ is exact at $E^{2,r}_{k,r}$, which is now sufficient to show that the sequence is exact.

Let $\beta = \beta_1\omega_1^+ + \beta_2\omega_2^+ + \beta_3\omega_3^+ \in E^{2,r+}_{k,r}$. Recall the lie in condition (19) that β must take the form $\beta = \frac{1}{r}\left(I(\beta_0)\omega_1^+ + J(\beta_0)\omega_2^+ + K(\beta_0)\omega_3^+\right)$ for some $\beta_0 \in E^0_{k-2,r}$. (The $\frac{1}{r}$-factor makes no difference here and is useful for cancellations.) Thus a general element of $E^{2,r}_{k,r}$ is of the form

$$\beta + \gamma = \frac{1}{r}\left(I(\beta_0)\omega_1^+ + J(\beta_0)\omega_2^+ + K(\beta_0)\omega_3^+\right) + \gamma_1\omega_1^- + \gamma_2\omega_2^- + \gamma_3\omega_3^-,$$

for $\beta_0, \gamma_j \in E^{0,r}_{k-2,r}$. A similar calculation to that of (23) shows that

$$\sigma(\beta+\gamma) = 0 \Longleftrightarrow \left\{\begin{array}{rcl} (r+2)\beta_0 + I(\gamma_1) + J(\gamma_2) + K(\gamma_3) &=& 0 \\ (r+2)\gamma_1 - I(\beta_0) - J(\gamma_3) + K(\gamma_2) &=& 0 \\ (r+2)\gamma_2 + I(\gamma_3) - J(\beta_0) - K(\gamma_1) &=& 0 \\ (r+2)\gamma_3 - I(\gamma_2) + J(\gamma_1) - K(\beta_0) &=& 0. \end{array}\right.$$

But this is exactly the lie in condition (17) which we need for $\beta_0 e^0 + \gamma_1 e^1 + \gamma_2 e^2 + \gamma_3 e^3$ to be in $E^{1,r}_{k-1,r-1}$, in which case we have

$$\beta + \gamma = \sigma\left(2(\beta_0 e^0 + \gamma_1 e^1 + \gamma_2 e^2 + \gamma_3 e^3)\right).$$

This demonstrates exactness at $E^{2,r}_{k,r}$ and so the middle sequence is exact. ∎

As a counterexample for the case $r = 0$ and $k \geq 4$, consider $\alpha \in E^0_{k-4,0}$. Then $\alpha e^{0123} \in E^{4,0}_{k,0}$ and $\sigma(\alpha e^{0123}) = 0$, so $\sigma : E_{k,0} \to E_{k+1,1}$ is not injective, which is exactly the same as saying that the symbol sequence is not exact at $E_{k,0}$. It is easy to see that this counterexample does not arise when $k = 0$ or 2, and to show that the maps $\sigma : E_{0,0} \to E_{1,1}$ and $\sigma : E_{2,0} \to E_{3,1}$ *are* injective.

As a counterexample for the case $r = 1$ and $k \geq 2$, consider $\alpha \in E^0_{k-2,0}$. Then $\alpha e^{123} \in E^{3,0}_{k+1,1}$ and $\alpha e^{123} \wedge e^0 \in E^4_{k+2,0}$. Thus $\sigma(\alpha e^{123}) = 0$. Since αe^{123} has no e^0-components at all it is clear that $\alpha e^{123} \neq \sigma(\beta)$ for any $\beta \in E^2_{k,0}$. Thus the symbol sequence fails to be exact at $E_{k+1,1}$. Again, it is easy to see that this counterexample does not arise when $k = 0$, and to show that the sequence $E_{0,0} \xrightarrow{\sigma} E_{1,1} \xrightarrow{\sigma} E_{2,2}$ *is* exact at $E_{1,1}$.

This concludes our proof of Theorem 5.1.

6 Quaternion-valued forms on Hypercomplex Manifolds

Let M be a hypercomplex manifold. Then M has a triple (I, J, K) of complex structures which generates the $\mathfrak{sp}(1)$-action on $\Lambda^k T^*M$ and which we can identify globally with the imaginary quaternions. Joyce has used this identification to define 'quaternionic holomorphic functions', which he calls q-holomorphic functions. A quaternion-valued function $f = f_0 + f_1 i + f_2 j + f_3 k$ is defined to be q-holomorphic if it satisfies a quaternionic version of the Cauchy-Riemann equations [J, 3.3]

$$df_0 + I(df_1) + J(df_2) + K(df_3) = 0. \tag{24}$$

This equation can also be obtained by comparing the Sp(1)-representations on T^*M and on the quaternions themselves. Recall that Equation (5) describes the $\mathrm{Sp}(1)\mathrm{GL}(n, \mathbb{H})$-representation on $\mathbb{H}^n$ as $V_1 \otimes E$. In the case $n = 1$ this reduces to the representation

$$\mathbb{H} \cong V_1 \otimes V_1, \tag{25}$$

interpreting the left-hand copy of V_1 as the left-action $(p, q) \mapsto pq$, and the right-hand copy of V_1 as the right-action $(p, q) \mapsto qp^{-1}$, for $q \in \mathbb{H}$ and $p \in \mathrm{Sp}(1)$.

We can now use our globally defined hypercomplex structure to combine the Sp(1)-actions on $\mathbb{H}$ and $\Lambda^k T^*M$. Consider, for example, quaternion-valued exterior forms in the bundle $E_{k,r} = \epsilon^n_{k,r} V_r$. The Sp(1)-action on these forms is described by the representation

$$\mathbb{H} \otimes E_{k,r} \cong V_1 \otimes V_1 \otimes \epsilon^n_{k,r} V_r.$$

Leaving the left $\mathbb{H}$-action untouched, we consider the effect of the right $\mathbb{H}$-action and the hypercomplex structure simultaneously. This amounts to applying the operators

$$\mathcal{I} : \alpha \to I(\alpha) - \alpha i, \qquad \mathcal{J} : \alpha \to J(\alpha) - \alpha j \qquad \text{and} \qquad \mathcal{K} : \alpha \to K(\alpha) - \alpha k$$

to $\alpha \in \mathbb{H} \otimes E_{k,r}$. Under this diagonal action the tensor product $V_1 \otimes \epsilon^n_{k,r} V_r$ splits, giving the representation

$$\mathbb{H} \otimes E_{k,r} \cong V_1 \otimes \epsilon^n_{k,r} (V_{r+1} \oplus V_{r-1}). \tag{26}$$

This gives a quaternion-valued version of the double complex which has certain advantages over its real-valued counterpart — for example, in 4-dimensions the whole quaternionic double complex is elliptic.

Joyce's equation (24) turns out to be one example of this: it is the condition necessary for df to lie in the V_2-summand of the splitting

$$\mathbb{H} \otimes T^*M \cong V_1 \otimes 2n(V_2 \oplus V_0).$$

Joyce's paper also develops a theory of quaternionic algebra based upon left $\mathbb{H}$-modules whose structure is 'augmented' by singling out a particular real subspace. The author has shown that the most important class of these 'augmented $\mathbb{H}$-modules' arises from Sp(1)-representations using splittings of the form given by Equation (26). This point of view turns out to be very fruitful: it both improves our understanding of Joyce's quaternionic algebra and shows how to apply his theory to many naturally occuring vector bundles over hypercomplex manifolds. This leads not only to Joyce's q-holomorphic functions, but also to quaternionic analogues of holomorphic k-forms, the holomorphic tangent and cotangent spaces, and even complex Lie groups and Lie algebras. The author hopes to make this type of quaternionic analysis on hypercomplex manifolds the subject of a future paper.

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